Statistics and Probability Letters 135 (2018) 38–43

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

The characterization of tenable Pólya urns

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ARTICLE INFO

Article history:
Received 15 August 2017
Received in revised form 14 October 2017
Accepted 29 November 2017
Available online 12 December 2017

Keywords:
Random structure
Pólya urn
Determinant

ABSTRACT

We characterize tenable Pólya urn schemes via a decomposition of their replacement matrices into small submatrices with certain conditions on the determinants. The characterization also involves an interplay between the submatrices and the initial conditions, as well as certain divisibility conditions.

1. Introduction

A Pólya urn scheme consists of a container that has τ_n balls present at the nth step, and each ball can be any of $k > 0$ possible colors. The color of an individual ball does not change. The dynamics of the system are attributed to the addition or removal of balls, into or out of the system. An urn scheme consists of four (repeated) steps:

1. Drawing a ball out of the urn, uniformly at random;
2. Noting the color i of the ball, where $i \in \{1, 2, \ldots, k\}$;
3. Placing the ball back into the urn;
4. Utilizing a set of rules for changing the quantities of balls in the urn, according to a replacement matrix $\mathbf{A}$, depending on the color of the selected ball.

This urn scheme has been well studied. The origins of the scheme can be traced back (at least) as far as 1977 in the works of Johnson and Kotz (1977). If the steps can be repeated indefinitely – without running out of all colors simultaneously, and without having insufficient numbers of balls for the required replacements – the urn scheme is called *tenable*. If at least one path exists in which the urn scheme cannot satisfy the replacement matrix rules, or in which the total number of balls reaches 0, then the urn scheme is *untenable*. A state in which the replacement matrix rules cannot successfully be applied is known as an *untenable state*.

The scheme begins with a vector $\mathbf{X}_0 = (X_{0,1}, \ldots, X_{0,k})$. The variable $X_{0,i}$ denotes the initial number of balls of color i in the urn. We use $\tau_0 := \sum_i X_{0,i}$ to denote the total number of balls in the urn at the start. The replacement matrix $\mathbf{A}$ is of size $k \times k$. The entry $A_{i,j}$ (on the ith row and jth column) denotes the number of balls of color j that are added to the urn, when a ball of color i is drawn (Mahmoud, 2003). All Pólya urn schemes are composed of two elements: the initial conditions, $\mathbf{X}_0$, and the replacement matrix, $\mathbf{A}$, written $(\mathbf{X}_0, \mathbf{A})$.

It is of interest to characterize and identify the unique starting conditions and replacement matrix rule conditions that will encompass all tenable urns. This paper characterizes such tenable urn schemes through mathematically necessary conditions

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https://doi.org/10.1016/j.spl.2017.11.013

for tenability. This characterization can quickly pinpoint if a model is tenable, where an untenable model is getting stuck, and how untenable models can be adjusted to become tenable. Furthermore, these results can be extended to find the smallest change to an untenable urn scheme that will result in a tenable urn, and lays the foundation for an if and only if statement for all tenable urns.

The Pólya–Eggenberger urn (Johnson and Kotz, 1977) is an example of a tenable urn scheme involving two colors, whereas the Pills Problem (Hwang et al., 2007) is an untenable urn scheme involving two colors. Mahmoud (1997) describes an urn model representing binary trees using three colors, and Mahmoud and Smythe (1992) use four different colored balls to represent the placement of a node inside a tree. Further, Yamato (1993), Holst (1979), Janson (2003), and Athreya and Karlin (1968) describe urn models using multiple colors. It is therefore of interest when studying urn models to not restrict the urn to two colors, but to generalize to an urn with up to k colors.

With the vast use of urn models in different disciplines, it is of interest to examine the structure of urn models. Using characteristics such as the entries in the replacement matrix, growth or decay over time, and number of balls added on each draw, many categories of urns have been studied. Balanced urns, in which the number of balls in the urn remains constant, have been very well-studied; see, for instance, Bagchi and Pal (1985), Bai and Hu (1999), Flajolet et al. (2005), Gouet (1989) and Mahmoud (2009). Many subcategories of urn models have been developed including: sacrificial urns (and semi-sacrificial urns) (Flajolet et al., 2006), elliptic urns (Flajolet et al., 2005), and triangular urns (Janson, 2006) (which also appears as part of Flajolet et al., 2005). Models have also been studied in which the number of balls in the urn is diminishing (Kuba and Panholzer, 2010).

Each of these current categorizations are defined only using the replacement matrix $\mathbf{A}$. There exists a much broader category of urns known as *tenable urn schemes*, that requires knowledge of both the replacement matrix $\mathbf{A}$ and the initial conditions $\mathbf{X}_0$ (Janson, 2003; Johnson and Kotz, 1977; Mahmoud, 2003, 2009). This paper looks to characterize all tenable urn schemes.

2. Tenable urns

We use $X_{n,i}$ to denote the number of balls of color i at the nth step. If $(\mathbf{X}_0, \mathbf{A})$ is an urn scheme, we need to have $X_{n,i} \geq 0$ for all n and i to maintain tenability, i.e., we can never have a negative number of balls of color i in the urn. Additionally, a tenable urn needs at least one ball present at all times. Therefore the number of balls τ_n that are in the urn at the nth step needs to satisfy $\tau_n := \sum_i X_{n,i} \geq 1$.

Suppose that an urn scheme $(\mathbf{X}_0, \mathbf{A})$ begins with at least one ball (i.e., $\tau_0 > 0$), and all elements of the replacement matrix are non-negative, i.e., $A_{i,j} \geq 0 \; \forall \, i, j$. The urn scheme $(\mathbf{X}_0, \mathbf{A})$ is tenable, as the number of balls for all colors is always non-decreasing, and therefore will never reach an untenable state. Therefore, tenability is only in question if at least one element in $\mathbf{A}$ is negative. For this reason, all characterizations that follow focus on negative values in $\mathbf{A}$ and conditions needed to maintain tenability in such instances. Specifically, there are three categorizations of negatives occurring in the replacement matrix: negative entries occurring on the diagonal elements (i.e., $A_{i,i}$), negative entries occurring in the same column (i.e., $A_{*,i}$), and negative entries occurring in the same row (i.e., $A_{i,*}$).

2.1. All essentiality

For a given urn scheme $(\mathbf{X}_0, \mathbf{A})$, if color i never appears in the urn, i.e., if $X_{n,i} = 0 \; \forall \, n$, then row i of the replacement matrix (i.e., $A_{i,*}$), becomes irrelevant. The rules of $A_{i,*}$ will only be enforced if a ball of color i is chosen from the urn. As no ball of color i ever exists in the urn, then $A_{i,*}$ does not affect the tenability of the urn scheme. In such a case, color i is referred to as *non-essential*. Conversely, a color is referred to as *essential* if the probability is 0 that a stochastic path will require taking out more balls of that color than are present in the urn, and the probability is 1 that it will appear in the urn at some point n.

All-essentiality of a tenable Pólya urn scheme $(\mathbf{X}_0, \mathbf{A})$ means that almost surely, any ball color will eventually appear while drawing. One can naturally decompose a k-color tenable scheme into smaller all-essential schemes. For this reason, *it would be adequate to derive all following theorems only for all-essential schemes*.

2.2. Negatives on the diagonal

The first instance of removing balls from the urn due to negative values in the replacement matrix that will be explored is the removal of balls of the same color as the ball drawn. In other words, a negative value on a diagonal element of the replacement matrix (i.e., $A_{i,i} < 0$).

Theorem 1 (*Gouet, 1989* and *Janson, 2003*)**.** *Let $(\mathbf{X}_0, \mathbf{A})$ be an all-essential tenable urn and suppose there exists an i such that $A_{i,i} < 0$. Then the starting value $X_{0,i}$ is a non-negative integer multiple of $|A_{i,i}|$, and each $A_{\ell,i}$ is an integer multiple of $|A_{i,i}|$.*

2.3. Negatives in the same column

The next instance of removing balls from the urn due to negative values in the replacement matrix involves the draw of a ball of color i requiring the removal of balls of color i from the urn, as well as the draw of a ball of color j requiring the removal of balls of color i. In the replacement matrix (i.e., $A_{i,j} < 0$).

Theorem 2. *Let* $(\mathbf{X}_0, \mathbf{A})$ *be an all-essential tenable urn scheme such that* $A_{i,i} < 0$ *and* $A_{j,i} < 0$*, then*

1. $A_{j,j} < 0$,
2. $A_{i,i} = A_{j,i}$ *and* $A_{i,j} = A_{j,j}$*, and*
3. $\det \begin{pmatrix} A_{i,i} & A_{i,j} \\ X_{0,i} & X_{0,j} \end{pmatrix} = 0$.
4. *If, in addition, there exists an* ℓ *such that* $A_{\ell,i} \geq 0$*, then* $\det \begin{pmatrix} A_{i,i} & A_{i,j} \\ A_{\ell,i} & A_{\ell,j} \end{pmatrix} = 0$.
5. *If, in addition, there exists an* ℓ *such that* $A_{\ell,i} < 0$*, then* $A_{\ell,j} = A_{j,j}$ *and* $A_{\ell,\ell} = A_{j,\ell}$.

Proof. Suppose that $A_{j,j}$ is nonnegative. As color j is essential, there exists an n such that $X_{n,j} > 0$. After $\lfloor \frac{X_{n,i}}{|A_{j,i}|} \rfloor + 1$ consecutive draws of a ball of color j, the urn will reach an untenable state. Therefore, $A_{j,j} < 0$.

Suppose that $A_{i,i} \neq A_{j,i}$ and without loss of generality, $|A_{j,i}| < |A_{i,i}|$. It is possible to draw $\frac{X_{n,j}}{|A_{j,j}|} - 1$ balls of color j, after which there will be $|A_{j,i}| < |A_{i,i}|$ balls of color i remaining in the urn. Upon drawing a ball of color i, the urn scheme will reach a state of untenability, as the drawing of such a ball requires the removal of $|A_{i,i}| > |A_{j,i}|$ balls of color i, while only $|A_{j,i}|$ remain. Therefore $A_{i,i} = A_{j,i}$, and a similar argument can be used to also show that $A_{i,j} = A_{j,j}$.

From Theorem 1, there exists m_i such that $X_{n,i} = m_i|A_{i,i}|$, and similarly there exists m_j such that $X_{n,j} = m_j|A_{j,j}|$. Since $A_{i,j} = A_{j,j}$, then $X_{n,j} = m_j|A_{i,j}|$. Suppose $m_i \neq m_j$, and without loss of generality, $m_i < m_j$. It is then possible to have m_i consecutive draws of balls of color i, after which all balls of color i would be depleted from the urn, however, not all balls of color j would be depleted from the urn, which has already been shown cannot occur for a tenable urn. Therefore, $m_i = \frac{\mathbf{X}_{n,i}}{|A_{i,i}|} = \frac{\mathbf{X}_{n,j}}{|A_{i,j}|} = m_j$, or equivalently, $\det \begin{pmatrix} A_{i,i} & A_{i,j} \\ \mathbf{X}_{0,i} & \mathbf{X}_{0,j} \end{pmatrix} = 0$.

If the draw of a ball of color ℓ adds balls of either color i or j, in other words either $A_{\ell,i} > 0$ or $A_{\ell,j} > 0$, the condition that balls of color i occur in the urn if and only if balls of color j occur in the urn must continue to hold. A similar argument can be used to show that $\det \begin{pmatrix} A_{i,i} & A_{i,j} \\ A_{\ell,i} & A_{\ell,j} \end{pmatrix} = 0$as was used to show $\det \begin{pmatrix} A_{i,i} & A_{i,j} \\ \mathbf{X}_{0,i} & \mathbf{X}_{0,j} \end{pmatrix} = 0$.

Suppose $A_{\ell,i} < 0$ and (without loss of generality) $|A_{\ell,j}| < |A_{j,j}|$. Then it is possible to draw balls of color ℓ consecutively until all balls of color ℓ and i are depleted from the urn, however, not depleting all balls of color j, as has been shown is not possible with a tenable urn scheme. Therefore, if $A_{\ell,i} < 0$, $A_{\ell,j} = A_{j,j}$ and a similar argument can be used to show $A_{\ell,\ell} = A_{j,\ell}$. □

2.4. Negatives in the same row

The last instance of negative values occurring in the replacement matrix include negative values that are in the same row.

Theorem 3. *Let* $(\mathbf{X}_0, \mathbf{A})$ *be an all-essential tenable urn scheme where* $A_{i,i} < 0$, $A_{i,j} < 0$*, and* $A_{j,i} \geq 0$*. In such a case,* $\det \begin{pmatrix} A_{i,i} & A_{i,j} \\ X_{n,i} & X_{n,j} \end{pmatrix} \leq 0$ *and* $\det \begin{pmatrix} A_{i,i} & A_{i,j} \\ A_{j,i} & A_{j,j} \end{pmatrix} \leq 0$.

Proof. The urn must start with enough balls of colors i and j such that if balls of color i are drawn consecutively, the urn will not be depleted of balls of color j prior to being depleted of balls of color i.

At any time n, it is possible to have $\frac{X_{n,i}}{|A_{i,i}|}$ consecutive draws of balls of color i, resulting in the removal of $\frac{X_{n,i}}{|A_{i,i}|}|A_{i,j}|$ balls of color j from the urn. If $X_{n,j} < \frac{X_{n,i}}{A_{i,i}}A_{i,j}$, then the urn will reach a state of untenability. Therefore, $X_{n,j} \geq \frac{X_{n,i}}{A_{i,i}}A_{i,j}$ which is equivalent to $\det \begin{pmatrix} A_{i,i} & A_{i,j} \\ X_{n,i} & X_{n,j} \end{pmatrix} \leq 0$.

A similar argument can then be used to show $A_{j,j} \geq \frac{A_{j,i}}{|A_{i,i}|}A_{i,j}$, which is equivalent to $\det \begin{pmatrix} A_{i,i} & A_{i,j} \\ A_{j,i} & A_{j,j} \end{pmatrix} \leq 0$. □

Theorem 4. *Let* $(\mathbf{X}_0, \mathbf{A})$ *be an all-essential tenable urn scheme where* $A_{i,i} < 0$, $A_{i,j} < 0$*, and* $A_{j,i} \geq 0$*. If* $A_{\ell,\ell} \geq 0$ *or if* $A_{\ell,\ell} < 0$ *but the product* $A_{i,\ell}A_{\ell,i} \neq 0$*, then* $\det \begin{pmatrix} A_{i,i} & A_{i,j} \\ A_{\ell,i} & A_{\ell,j} \end{pmatrix} \leq 0$.

Proof. A draw of a ball of color ℓ provides the possibility of an additional $\frac{A_{\ell,i}}{|A_{i,i}|}$ consecutive draws of a ball of color i from the urn, which in turn would remove $\frac{A_{\ell,i}}{|A_{i,i}|}|A_{i,j}|$ balls of color j from the urn. Therefore $A_{\ell,j} \geq \frac{A_{\ell,i}}{|A_{i,i}|}|A_{i,j}|$, which is equivalent to $\det \begin{pmatrix} A_{i,i} & A_{i,j} \\ A_{\ell,i} & A_{\ell,j} \end{pmatrix} \leq 0$. □

The conditions of Theorem 4 may at first seem extraneous; however, additional urn schemes are also tenable when $A_{\ell,\ell} < 0$ and $A_{i,\ell}A_{\ell,i} = 0$, but where $\det \begin{pmatrix} A_{i,i} & A_{i,j} \\ A_{\ell,i} & A_{\ell,j} \end{pmatrix} > 0$ such as the following urn scheme:

$$(\mathbf{X}_0, \mathbf{A}) = \left(\begin{pmatrix} W_0 = 1 \\ B_0 = 3 \\ R_0 = 1 \end{pmatrix}, \begin{pmatrix} -1 & -1 & 0 \\ 3 & 4 & 0 \\ 1 & 0 & -1 \end{pmatrix} \right).$$

Therefore, additional requirements must be specified to account for such urn schemes.

Theorem 5. *Let* $(\mathbf{X}_0, \mathbf{A})$ *be an all-essential tenable urn scheme where* $A_{i,i} < 0$, $A_{i,j} < 0$, *and* $A_{j,i} \geq 0$. *If* $A_{\ell,\ell} < 0$, $A_{i,\ell}A_{\ell,i} = 0$, *and* $\det\begin{pmatrix} A_{i,i} & A_{i,j} \\ A_{\ell,i} & A_{\ell,j} \end{pmatrix} > 0$, *then* $\det\begin{pmatrix} A_{i,i} & A_{i,j} & A_{i,\ell} \\ A_{j,i} & A_{j,j} & A_{j,\ell} \\ A_{\ell,i} & A_{\ell,j} & A_{\ell,\ell} \end{pmatrix} \geq 0$ *and* $\det\begin{pmatrix} A_{i,i} & A_{i,j} & A_{i,\ell} \\ X_{n,i} & X_{n,j} & X_{n,\ell} \\ A_{\ell,i} & A_{\ell,j} & A_{\ell,\ell} \end{pmatrix} \geq 0$.

Proof. As $A_{i,i} < 0$, drawing consecutive balls of color i will eventually deplete balls of color i from the urn. The same is true for color ℓ, as $A_{\ell,\ell} < 0$.

Focusing only on balls of color i and ℓ for the moment, if $A_{i,\ell}A_{\ell,i} \neq 0$, (both quantities being positive), it would be possible to draw balls of colors i and ℓ ad infimum, as a draw of one color would add at least one ball of the other color, preventing the urn from being depleted of both colors simultaneously. If $A_{i,\ell}A_{\ell,i} = 0$, there remains a finite number of consecutive draws of colors i and ℓ possible before all balls of colors i and ℓ are removed from the urn.

Expanding the scope to look at balls of color j, a draw of a ball of color i requires the removal of balls of color j, and a draw of a ball of color ℓ from the urn has the potential to add more balls of color i to the urn. The goal is to make sure that after all balls of colors i and ℓ have been removed from the urn, starting at time n, the total number of balls of color j removed (call this quantity $\mathcal{R}$) does not exceed the total number of balls of color j remaining in the urn (call this quantity $\mathcal{J}$). The quantity $\mathcal{J}$ consists of the initial number of balls of color j in the urn at time n, $\mathbf{X}_{n,j}$, in addition to the following two quantities.

1. The number of balls of color j added to the urn if $\frac{X_{n,\ell}}{|A_{\ell,\ell}|}$ balls of color ℓ are drawn consecutively from time n until all balls of color ℓ are removed from the urn, $\frac{X_{n,\ell}}{|A_{\ell,\ell}|}A_{\ell,j}$.
2. The number of balls of color j added to the urn after all balls of color i are removed from the urn by consecutively drawing balls of color i from the urn, (which could add additional balls of color ℓ into the urn), and then all subsequent balls of color ℓ are removed by consecutively drawing balls of color ℓ, $\left(\frac{X_{n,i}}{|A_{i,i}|}\frac{A_{i,\ell}}{|A_{\ell,\ell}|}\right)A_{\ell,j}$.

Thus, $\mathcal{J} = X_{n,j} + \left(\frac{X_{n,\ell}}{|A_{\ell,\ell}|} + \frac{X_{n,i}}{|A_{i,i}|}\frac{A_{i,\ell}}{|A_{\ell,\ell}|}\right)A_{\ell,j}$.

The quantity $\mathcal{R}$ is composed of two parts:

1. The number of balls of color j removed after all balls of color i are removed from the urn by consecutively drawing balls of color i from the urn, $\frac{X_{n,i}}{|A_{i,i}|}|A_{i,j}|$.
2. The number of balls of color j removed after all balls of color ℓ are removed from the urn by consecutively drawing balls of color ℓ from the urn, (which could add additional balls of color i into the urn), and then all subsequent balls of color i are removed by consecutively drawing balls of color i, $\left(\frac{X_{n,\ell}}{|A_{\ell,\ell}|}\frac{A_{\ell,i}}{|A_{i,i}|}\right)|A_{i,j}|$.

Thus, $\mathcal{R} = \left(\frac{X_{n,i}}{|A_{i,i}|} + \frac{X_{n,\ell}}{|A_{\ell,\ell}|}\frac{A_{\ell,i}}{|A_{i,i}|}\right)|A_{i,j}|$.

Therefore the following equation must be satisfied:

$$\left(\frac{X_{n,i}}{|A_{i,i}|} + \frac{X_{n,\ell}}{|A_{\ell,\ell}|}\frac{A_{\ell,i}}{|A_{i,i}|}\right)|A_{i,j}| \leq X_{n,j} + \left(\frac{X_{n,\ell}}{|A_{\ell,\ell}|} + \frac{X_{n,i}}{|A_{i,i}|}\frac{A_{i,\ell}}{|A_{\ell,\ell}|}\right)A_{\ell,j}.$$

This is equivalent to

$$\det\begin{pmatrix} A_{i,i} & A_{i,j} & A_{i,\ell} \\ X_{n,i} & X_{n,j} & X_{n,\ell} \\ A_{\ell,i} & A_{\ell,j} & A_{\ell,\ell} \end{pmatrix} \geq 0.$$

The same argument follows for the values of the replacement matrix $A_{j,i}, A_{j,j}, A_{j,\ell}$, replacing $X_{n,i}, X_{n,j}, X_{n,\ell}$, respectively. □

Suppose there exists an urn scheme for which the conditions of Theorem 5 are met, but a draw of a ball of color m adds additional balls of color ℓ to the urn, $A_{m,\ell} > 0$. The theorem that follows outlines characteristics of a tenable urn scheme where it is possible for the urn to always have at least one ball of either color i, ℓ, or m. In other words, there exists at least one stochastic path for which the urn is never void of all three colors (i, ℓ, and m) of balls simultaneously.

Theorem 6. *Let* $(\mathbf{X}_0, \mathbf{A})$ *be an all-essential tenable urn scheme where* $A_{i,i} < 0$, $A_{i,j} < 0$, *and* $A_{j,i} \geq 0$. *In addition, let* $A_{\ell,\ell} < 0$, $A_{i,\ell}A_{\ell,i} = 0$, *and* $\det\begin{pmatrix} A_{i,i} & A_{i,j} \\ A_{\ell,i} & A_{\ell,j} \end{pmatrix} > 0$. *If* $A_{m,\ell} > 0$, *and* $A_{m,\mathrm{m}} > 0$, *or* $A_{m,\mathrm{m}} < 0$ *but* $A_{\ell,\mathrm{m}} > 0$, *or* $A_{m,\mathrm{m}} < 0$ *but* $A_{i,\mathrm{m}} > 0$, *then* $\det\begin{pmatrix} A_{i,i} & A_{i,j} & A_{i,\ell} \\ A_{\ell,i} & A_{\ell,j} & A_{\ell,\ell} \\ A_{m,i} & A_{m,j} & A_{m,\ell} \end{pmatrix} \leq 0$.

Proof. In the scenario described, a draw of a ball of color i or ℓ has the same impact on the number of balls of color j in the urn as seen in Theorem 5, with the added complexity of an additional ball color, m, that allows balls of color i or ℓ to be drawn ad infimum. Either color m always exists in the urn, or a draw of color ℓ adds additional balls of color m into the urn, or a draw of a ball of color i adds additional balls of color m to the urn. Therefore, each draw of a ball of color m, must add enough balls of color j to counteract balls of color j that are removed from drawing balls of colors i and ℓ.

The total number of additional balls of color j that are removed from the urn by drawing balls of colors i or ℓ that have been added to the urn by a draw of a ball of color m, id denoted by the quantity $\mathcal{R} = \left(\frac{A_{m,i}}{|A_{i,i}|} + \frac{A_{m,\ell}}{|A_{\ell,\ell}|}\frac{A_{\ell,i}}{|A_{i,i}|}\right)|A_{i,j}|$.

The total number of balls of color j that are added to the urn as a result of drawing a ball of color m, and then any additional balls of color j that are added after consecutively drawing all balls of color ℓ that have been added as a result of drawing a ball of color m, is denoted by the quantity $\mathcal{J} = A_{m,j} + \frac{A_{m,\ell}}{|A_{\ell,\ell}|}A_{\ell,j}$.

To be tenable, we need $\mathcal{R} \leq \mathcal{J}$, which yields:

$$\left(\frac{A_{m,i}}{|A_{i,i}|} + \frac{A_{m,\ell}}{|A_{\ell,\ell}|}\frac{A_{\ell,i}}{|A_{i,i}|}\right)|A_{i,j}| \leq A_{m,j} + \frac{A_{m,\ell}}{|A_{\ell,\ell}|}A_{\ell,j}.$$

The inequality can be expressed as:

$$A_{m,i}A_{i,j}A_{\ell,\ell} - A_{m,\ell}A_{\ell,i}A_{i,j} - A_{m,j}A_{i,i}A_{\ell,\ell} + A_{m,\ell}A_{\ell,j}A_{i,i} \leq 0$$

and therefore

$$\det\begin{pmatrix} A_{i,i} & A_{i,j} & A_{i,\ell} \\ A_{\ell,i} & A_{\ell,j} & A_{\ell,\ell} \\ A_{m,i} & A_{m,j} & A_{m,\ell} \end{pmatrix} \leq 0. \quad \square$$

There exists a specific case that is not covered in Theorem 6, and which acts in the same way as the case specified in Theorem 5. In the case referred to in Theorem 5, the urn was eventually void of colors i and ℓ simultaneously. In a similar manner, the following example will eventually be depleted of colors i, ℓ, and m simultaneously, providing a case not covered by Theorem 6.

$$(\mathbf{X}_0, \mathbf{A}) = \left(\begin{pmatrix} W_0 = 1 \\ B_0 = 3 \\ R_0 = 1 \\ Y_0 = 1 \end{pmatrix}, \begin{pmatrix} -1 & -1 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 1 & -1 \end{pmatrix}\right).$$

Thus, Theorem 5 can be generalized to any sequence of colors $\mathcal{S}$ for which drawing balls of these colors (in any order) will eventually add additional balls of color i into the urn, where the consecutive drawing of the sequence of colors will also result in the simultaneous removal of all colors in said sequence.

Without loss of generality, the urn scheme $(\mathbf{X}_0, \mathbf{A})$ can be ordered in such a way that $i = 1, j = 2$, and $\mathcal{S} \subseteq \{3, 4, \ldots, k\}$, where there are s colors included in the subset $\mathcal{S}$. Additionally, let $\mathbf{X}(\{i, j, \mathcal{S}\})$ be the elements $X_{n,i}, X_{n,j}, X_{n,s_1}$, and so on. This notation will be used to simplify the following determinant inequality:

$$\det\begin{pmatrix} A_{i,i} & A_{i,j} & A_{i,\ell} \\ X_{n,i} & X_{n,j} & X_{n,\ell} \\ A_{\ell,i} & A_{\ell,j} & A_{\ell,\ell} \end{pmatrix} \geq 0 \qquad \text{is equivalent to } \det(\mathbf{A}(\{i, \mathbf{X}(\{i, j, \ell\}), \ell\})) \geq 0.$$

Theorem 7. *Let $(\mathbf{X}_0, \mathbf{A})$ be an all-essential tenable urn scheme where $A_{i,i} < 0$, $A_{j,i} < 0$, and $A_{j,i} \geq 0$. If there exists $s_i \in \mathcal{S}$ such that $A_{s_i,i} > 0$, and $\forall\, s_i \in \mathcal{S}$, $A_{s_i,s_i} < 0$, $A_{s_i,s_j} * A_{s_j,s_i} = 0$, and $A_{i,s_i} = 0$ then the following inequalities hold:*

- $(-1)^{s+2}\det(\mathbf{A}\{i, j, \mathcal{S}\}) \leq 0$,
- $(-1)^{s+2}\det(\mathbf{A}\{i, \mathbf{X}(\{i, j, \mathcal{S}\}), \mathcal{S}\}) \leq 0$.

The details of this proof are left to the reader, as they mirror earlier arguments.

3. Sufficient conditions and future research

The theorems outlined in this paper are the necessary conditions for tenability and assume an urn is tenable, therefore it cannot reach an empty state. The sufficient conditions will therefore require not only the theorems outlined in this paper, but also additional conditions that will prevent the urn from reaching an empty state.

An example illustrating this point is the urn scheme $(\mathbf{X}_0, \mathbf{A}) = \left(\begin{pmatrix} W_0 = 2 \\ B_0 = 0 \end{pmatrix}, \begin{pmatrix} -2 & 0 \\ 2 & 0 \end{pmatrix}\right)$. This urn scheme does not violate Theorem 1, yet is not tenable as after the first draw of a ball and the conditions that follow, the urn scheme is rendered empty.

For an urn to always have at least one ball, the urn scheme must have at least one essential color. One must therefore determine the characteristics of $(\mathbf{X}_0, \mathbf{A})$ required for a color to be considered essential. In general, color i is essential if $A_{i,i} \geq 0$ and either $\mathbf{X}_{0,i} > 0$ or $\exists\, j \mid A_{j,i} > 0$ where $\mathbf{X}_{j,n} > 0$ for some time n. If $A_{i,i} < 0$ color i is essential if $\exists\, j \mid A_{j,i} > 0$ where color j is an essential color.

Additionally, there are certain rows in $\mathbf{A}$ that need not follow the conditions laid out in this paper; namely, a color m that never appears in the urn. If color m never appears in the urn, in order for the urn to be tenable, $\mathbf{X}_{n,\,\mathrm{m}} = 0$ for all n and $A_{i,\,\mathrm{m}} = 0$ for all i such that $\mathbf{X}_{i,n} > 0$ for some time n. All values of $A_{m,*}$ and $A_{j,\,\mathrm{m}}$ for all j that also never appear in the urn, need not follow the conditions laid out in this paper, as these conditions will never be invoked.

For brevity of this paper, additional details regarding specific sufficient conditions are left for future research.

Acknowledgments

Both authors thank Professor Hosam M. Mahmoud for guidance throughout this investigation. A. Davidson gratefully acknowledges Professor Mahmoud's mentoring and advice as a Co-Chair of her Ph.D. committee. M.D. Ward's research is supported by NSF DMS-#1246818 and by the NSF Science & Technology Center for Science of Information Grant CCF-0939370.

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